

ON THE IDEALS OF SECANT VARIETIES OF SEGRE VARIETIES

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ABSTRACT. We establish basic techniques for studying the ideals of secant varieties of Segre varieties. We solve a conjecture of Garcia, Stillman and Sturmfels on the generators of the ideal of the first secant variety in the case of three factors and solve the conjecture set-theoretically for an arbitrary number of factors. We determine the low degree components of the ideals of secant varieties of small dimension in a few cases.

1. INTRODUCTION

Let $X^n \subset \mathbb{P}V$ be a projective variety. Define $\sigma_r(X)$, the *variety of secant \mathbb{P}^{r-1} 's to X* by

$$\sigma_r(X) = \overline{\bigcup_{x_1, \dots, x_r \in X} \mathbb{P}_{x_1, \dots, x_r}}$$

where $\mathbb{P}_{x_1, \dots, x_r} \subset \mathbb{P}V$ denotes the linear space spanned by x_1, \dots, x_r (usually a \mathbb{P}^{r-1}).

Given $X \subset \mathbb{P}V$ and $p \in \mathbb{P}V$ define the *essential X -rank of p* (or *essential rank of p* if X is understood) to be the smallest r such that $p \in \sigma_r(X)$. (The essential rank is often called the *border rank* in the computational complexity literature.) Similarly define the *rank* of p to be the smallest r such that there exist r points on X , x_1, \dots, x_r such that $p \in \mathbb{P}_{x_1, \dots, x_r}$. The essential rank can be smaller than the rank, this phenomenon occurs already for $X = v_3(\mathbb{P}^1)$, the cubic curve, where the essential rank of any point is at most two, but the rank of points on a tangent line to X (but not on X) is three.

Given vector spaces A_1, \dots, A_k , one can form the *Segre product* $X = \text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k) \subset \mathbb{P}(A_1 \otimes \dots \otimes A_k)$. When $k = 2$, the Segre product is just the projectivization of the space of rank one elements (matrices) in $A_1 \otimes A_2$. In this paper we study the ideals of the varieties $\sigma_p(X)$. The case $k = 3$ is important in the study of computational complexity as explained below. Many cases are important in the study of Bayesian networks, as explained in [6]. After presenting some background information in §2, we establish basic techniques for studying the problem for an arbitrary rational homogeneous variety X in §3. In §4 we specialize to Segre products and take advantage of Schur duality. We determine $I_3(\sigma_2(X))$ for any Segre product in Theorem 4.7. We prove that $I_3(\sigma_2(X))$ cuts out $\sigma_2(X)$ set-theoretically in all cases and ideal theoretically when $k = 3$, partially resolving Conjecture 21 of [6], see Theorem 5.1.

We present a deterministic algorithm to find generators of the ideals of secant varieties of Segre varieties in §4. We carry this algorithm out in low degrees in §6. In particular, we show there are no equations in the ideal of $\sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ in degree less than nine. We plan to study higher degrees in a future paper.

One motivation for this paper is the following question in computational complexity: Let $A = (\mathbb{C}^n \otimes \mathbb{C}^m)^*$, $B = (\mathbb{C}^m \otimes \mathbb{C}^p)^*$, $C = \mathbb{C}^n \otimes \mathbb{C}^p$. The matrix multiplication operator M_{nmp} is an element of $A \otimes B \otimes C$. In standard coordinates M_{nmp} is the sum of nmp monomials. However, already if one takes $m = n = p = 2$, it is known that $\sigma_7(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3) = \mathbb{P}^{15}$ so the essential rank of M_{222} is at most seven. Strassen showed [13] that in fact the rank is at most seven by exhibiting an explicit expression of M_{222} as the sum of seven monomials, and moreover,

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he proved that the essential rank is at least six by a specialization argument. The rank of M_{222} was then shown to be seven in [15].

To determine the essential rank of a point $p \in \mathbb{P}V$ it is sufficient to find equation cutting out the varieties $\sigma_k(X)$ set theoretically and then to evaluate the polynomials at p . Thus once one finds equations of $\sigma_6(\text{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$ one can determine the essential rank of M_{222} .

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2. DIMENSIONS OF SECANT VARIETIES, ESPECIALLY HOMOGENOUS ONES

Let $X \subset \mathbb{P}V$ be a projective variety. An important fact about $\sigma_r(X)$ is Terracini's lemma, which implies that if (x_1, \dots, x_r) is a general point of $X \times X \times \dots \times X$ then

$$\hat{T}_{[\vec{x}_1 + \dots + \vec{x}_r]} \sigma_r(X) = \hat{T}_{x_1} X + \dots + \hat{T}_{x_r} X$$

where $\vec{x} \in V$ denotes a point in the line $\hat{x} \subset V$ corresponding to the point $x \in \mathbb{P}V$ and $\hat{T}_p Y \subset V$ denotes the affine tangent space to Y at p , the cone over the embedded tangent projective space $\tilde{T}_p Y \subset \mathbb{P}V$.

If X^n is smooth then the dimension of $\sigma_2(X)$ can be determined by taking three derivatives at a general point $x \in X$, see [7]. In particular, if the third fundamental form of X at x , $III_{X,x}$, is non-zero, then $\sigma_2(X)$ is of the expected dimension $2n + 1$.

The third fundamental form calculation immediately implies that all homogeneously embedded rational homogeneous varieties have $\sigma_2(X)$ of dimension $2\dim X + 1$ except for the following varieties (embeddings are the minimal homogeneous ones unless otherwise specified and if varieties occur more than one way we only list them once): $G(2, n) = A_{n-1}/P_2$ the Grassmanian of 2-planes through the origin in \mathbb{C}^n , $Q^{2n-1} = D_n/P_1$, $Q^{2n-2} = B_n/P_1$, the quadric hypersurfaces, $G_Q(2, 2n) = D_n/P_2$, $G_Q(2, 2n + 1) = B_n/P_2$, the Grassmanians of 2-planes through the origin isotropic for a quadratic form, $v_2(\mathbb{P}^n)$ the quadratically embedded Veronese, $G_\omega(2, 2n) = C_n/P_2$, the the Grassmanians of 2-planes through the origin isotropic for a symplectic form, F_4/P_4 , G_2/P_1 , E_6/P_1 , E_6/P_2 , E_7/P_1 , E_7/P_7 , E_8/P_8 . $\text{Seg}(\mathbb{P}^k \times \mathbb{P}^l) = A_k/P_1 \times A_l/P_1$. Here we use the ordering of the roots as in [2].

In all other cases the third fundamental form is easily seen to be nonzero, see [9]. In particular, for all triple and higher Segre products, $\sigma_2(X)$ is nondegenerate.

In fact Lickteig and Strassen [11, 12] show many triple Segre products have all secant varieties nondegenerate, in particular for $\text{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$ the filling secant variety σ_r is the expected number $r = \lceil n^3/(3n - 2) \rceil$ when $n > 3$. In particular, for $n = m^2$ we get roughly $m^4/3$ which is significantly greater than m^3 , so in higher dimensions matrix multiplication is far from being a generic tensor. (Lickteig's proof is very simple and elegant - one first observes that certain small cases, e.g., $\sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n)$ fills and then one reduces to such cases by writing a larger vector space as a sum of two dimensional spaces.)

3. IDEALS OF SECANT VARIETIES, ESPECIALLY HOMOGENEOUS ONES

For $A \subset S^k V^*$ define $A^{(p)} = (A \otimes S^p V^*) \cap S^{p+k} V^*$, the p -th prolongation of A . Let

$$\text{Base}(A) = \{[v] \in \mathbb{P}V \mid P(v) = 0 \ \forall P \in A\}.$$

Given a variety $Z \subset \mathbb{P}V$ we let $I(Z) \subset S^\bullet V^*$ denote its ideal and $I_d(Z) = I(Z) \cap S^d V^*$. We recall from [9] that ideals of secant varieties satisfy the *prolongation property*:

Lemma 3.1 ([9], Lemma 2.2). *Let $A \subset S^2 V^*$ be a system of quadrics with base locus $\text{Base}(A) \subset \mathbb{P}V$. Then*

$$\text{Base}(A^{(k-1)}) \supseteq \sigma_k(\text{Base}(A)).$$

Moreover, if $\text{Base}(A)$ is linearly non-degenerate, then for $k \geq 2$, $I_k(\sigma_k(\text{Base}(A))) = 0$, and if $A = I_2(\text{Base}(A))$, then $I_{k+1}(\sigma_k(\text{Base}(A))) = A^{(k)}$.

Geometrically, $A^{(k-1)} \supseteq \{\frac{\partial P}{\partial v} \mid P \in A^{(k)}\} = I_{(\text{Base } A^{(k)})_{\text{sing}}}$ and $\sigma_{k-1}(X) \subseteq (\sigma_k(X))_{\text{sing}}$.

Usually $I(\sigma_k(X))$ is not generated in degree $k+1$. For example, consider the simplest intersection of quadrics, four points in \mathbb{P}^2 . They generate six lines so $\sigma(X)$ is a hypersurface of degree six.

Corollary 3.2. *Let $X \subset \mathbb{P}V$ be a variety with $I(X)$ generated in degree d . Then for all $k \geq 0$, $I_{d+k-2}(\sigma_k(X)) = 0$.*

Given a variety $X \subset \mathbb{P}V$, a polynomial $P \in S^d V^*$, $d > k$, is in $I_d(\sigma_k(X))$ if and only if for any sequence of non negative integers m_1, \dots, m_k , with $m_1 + \dots + m_k = d$, we have $P(v_1^{m_1}, v_2^{m_2}, \dots, v_k^{m_k}) = 0$ for all $v_i \in \hat{X}$, where $v_i^{m_i} = v_i \circ \dots \circ v_i$ (m_i times). Here we interpret $P(v_1^{m_1}, v_2^{m_2}, \dots, v_k^{m_k})$ as the result of the successive contractions of P by the tensors $v_i^{m_i}$.

Now consider the case where $X = G/P \subset \mathbb{P}V_l$ is a homogeneously embedded rational homogeneous variety, i.e., the orbit of a highest weight line.

By an unpublished theorem of Kostant, $I_2(X) = (V_{2l})^\perp \subset S^2 V^*$ and $I(X)$ is generated in degree two. More generally, $I_k(X) = (V_{kl})^\perp \subset S^k V^*$. We adopt the notation that if $V = V_l$, we write $V^k = V_{kl}$.

Note that if $W \subset S^d V^*$ is an irreducible module, either all of W or none of it is in $I_d(\sigma_k(X))$. These remarks imply:

Proposition 3.3. *Let $X \subset \mathbb{P}V$ be a rational homogenous variety. Then a module $W \subset S^d V^*$ is in $I_d(\sigma_k(X))$ if and only if for all integers (a_1, \dots, a_p) such that $a_1 + 2a_2 + \dots + pa_p = d$ and $a_1 + \dots + a_p = k$, the contraction map*

$$(1) \quad W \otimes S^{a_1}(V) \otimes S^{a_2}(V^2) \otimes \dots \otimes S^{a_p}(V^p) \longrightarrow \mathbb{C}$$

is zero.

Corollary 3.4. *Let $X = G/P \subset \mathbb{P}V$ be a rational homogeneous variety. Then for all $d > 0$,*

- (1) $I_d(\sigma_d(X)) = 0$,
- (2) $I_{d+1}(\sigma_d(X))$ is the kernel of the contraction map $V^2 \otimes S^{d+1} V^* \rightarrow S^{d-1} V^*$,
- (3) let W be an irreducible component of $S^d V^*$, and suppose that for all integers (a_1, \dots, a_p) such that (a_1, \dots, a_p) such that $a_1 + 2a_2 + \dots + pa_p = d$ and $a_1 + \dots + a_p = k$, W^* is not an irreducible component of $S^{a_1}(V) \otimes S^{a_2}(V^2) \otimes \dots \otimes S^{a_p}(V^p)$. Then $W \subset I_d(\sigma_k(X))$.

Proof. (1) follows immediately from 3.2, (2) from the remarks about the ideals of homogeneous varieties and 3.1. (3) follows from (1) and Schur's lemma because if an irreducible submodule $W \subset S^d V^*$ does not belong to $I_d(\sigma_k(X))$, one of the contraction maps (1) must be non-zero. \square

Question. *Is the ideal of the first secant variety $\sigma_2(X)$ of a rational homogeneous variety $X = G/P \subset \mathbb{P}V$, generated by cubics (assuming its nonempty)? More generally, when is the ideal of $\sigma_d(X)$ generated in degree $d+1$?*

If $X = G/P \subset \mathbb{P}V$ is a Scorza variety, that is the set of rank one elements in a simple Jordan algebra \mathcal{J} (see [16]), then $I_k(\sigma_{k-1}(X))$ is uniformly described as the $k \times k$ minors in \mathcal{J} and it generates $I(\sigma_{k-1}(X))$. More generally, if $X = G/P$ is a sub-minuscule variety, that is, the set of tangent directions to lines through a point of a compact Hermitian symmetric space (see [9]), then $I(\sigma_{k-1}(X))$ is generated in degree k and there is a uniform description of I_k , see [9].

Let \mathfrak{g} be a complex simple Lie algebra and consider the rational homogeneous variety $X_{ad} \subset \mathbb{P}\mathfrak{g}$, the unique closed orbit of the corresponding adjoint Lie group. Then there are universal modules $Y'_k \subset S^k \mathfrak{g}^*$, see [10] and $Y'_k \subseteq I_k(\sigma_{k-1}(X_{ad}))$.

However, consider $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^6) \subset \mathbb{P}^{104}$. The expected dimension of $\sigma_8(X)$ is $8 \times 12 + 7 = 103$, thus if it is nondegenerate, $\sigma_8(X)$ is an invariant hypersurface. If its degree is d , $S^d(A \otimes B \otimes C)$ must contain a one dimensional irreducible factor $(\det A)^\alpha \otimes (\det B)^\beta \otimes (\det C)^\gamma$. In particular d is divisible by 3, 5 and 7. Conclusion: either $\sigma_8(X)$ is degenerate, or it is a hypersurface of degree a multiple of 35. This suggests that the degrees of the equations of the $\sigma_k(X)$ must be much larger than k .

4. SCHUR DUALITY AND EQUATIONS OF SEGRE PRODUCTS

Let A_1, \dots, A_k be vector spaces and let $V = A_1 \otimes \dots \otimes A_k$. In order to determine the ideals of secant varieties of Segre varieties $\text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k) \subset \mathbb{P}(A_1 \otimes \dots \otimes A_k)$, we need to understand the decomposition of $S^d V^*$ into irreducible modules. We begin by reviewing Schur duality (see, e.g., [5] for an introduction to Schur duality):

The irreducible representations of the symmetric group \mathfrak{S}_m are parametrized by the partitions of m . If π is such a partition, we let $[\pi]$ denote the corresponding \mathfrak{S}_m -module. For any vector space V , there is a natural action of \mathfrak{S}_m on $V^{\otimes m}$ and we define $S_\pi V$ the π -th Schur power of V by

$$S_\pi V := \text{Hom}_{\mathfrak{S}_m}([\pi], V^{\otimes m}),$$

the \mathfrak{S}_m -equivariant linear maps from $[\pi]$ to $V^{\otimes m}$. $S_\pi V$ is zero if π has more parts than the dimension of V , otherwise $S_\pi V$ is an irreducible $GL(V)$ -module. Schur duality is the assertion that the tautological map

$$\bigoplus_{|\pi|=m} [\pi] \otimes S_\pi V \longrightarrow V^{\otimes m}$$

is an isomorphism.

For example, the trivial representation $[m]$ gives rise to $S_m V = S^m V$ and the sign representation $[1, \dots, 1]$ gives rise to $S_{1, \dots, 1} V = \Lambda^m V$.

Proposition 4.1. *Let A_1, \dots, A_k be vector spaces. Then*

$$S^m(A_1 \otimes \dots \otimes A_k) = \bigoplus_{|\pi_1|=\dots=|\pi_k|=m} ([\pi_1] \otimes \dots \otimes [\pi_k])^{\mathfrak{S}_m} S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k,$$

where $([\pi_1] \otimes \dots \otimes [\pi_k])^{\mathfrak{S}_m}$ denotes the space of \mathfrak{S}_m -invariants (i.e., instances of the trivial representation) in the tensor product.

Proof. Apply Schur duality separately to each of A_1, \dots, A_k , take the tensor product of the corresponding isomorphisms, and compare with Schur duality for $A_1 \otimes \dots \otimes A_k$. \square

Note that, since the representations of \mathfrak{S}_m are self-dual, the dimension of $([\pi_1] \otimes \dots \otimes [\pi_k])^{\mathfrak{S}_m}$ is equal to the multiplicity of $[\pi_k]$ in the tensor product $[\pi_1] \otimes \dots \otimes [\pi_{k-1}]$. There is no general rule to compute such multiplicities, but for small m we can compute them using elementary character theory: if χ_π is the character of $[\pi]$, then

$$(2) \quad \dim ([\pi_1] \otimes \dots \otimes [\pi_k])^{\mathfrak{S}_m} = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \chi_{\pi_1}(\sigma) \cdots \chi_{\pi_k}(\sigma).$$

Proposition 4.2. *We have the following decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)$ into irreducible $GL(A_1) \times \cdots \times GL(A_k)$ -modules:*

$$S^3(A_1 \otimes \cdots \otimes A_k) = \bigoplus_{\substack{|I|+|J|+|L|=k, \\ |J|>1}} \frac{2^{j-1} - (-1)^{j-1}}{3} S_3 A_I \otimes S_{21} A_J \otimes S_{111} A_L \oplus \bigoplus_{\substack{|I|+|L|=k, \\ |L| \text{ even}}} S_3 A_I \otimes S_{111} A_L,$$

where I, J, L are multi-indices whose union is $1, \dots, k$, and we use the notation $S_\pi A_I = \otimes_{i \in I} S_\pi A_i$.

In particular, $S^3(A \otimes B \otimes C) = S_3 S_3 S_3 \oplus S_3 S_{21} S_{21} \oplus S_3 S_{111} S_{111} \oplus S_{21} S_{21} S_{21} \oplus S_{21} S_{21} S_{111}$ and thus is multiplicity free. Here $S_l S_\mu S_\nu$ is to be read as $S_l A \otimes S_\mu B \otimes S_\nu C$ plus permutations giving rise to distinct modules.

Proof. The irreducible representations of \mathfrak{S}_3 are the trivial representation [3], the sign representation [111], and the natural two-dimensional representation [21]. So we just need to compute the decomposition of $[21]^{\otimes j}$ into irreducible components, which is a simple character computation.

The symmetric group \mathfrak{S}_3 has three conjugacy classes of cardinality 1, 3, 2, and the values of the irreducible characters on these classes are given by the following table:

class	$[Id]$	$[(12)]$	$[(123)]$
#	1	3	2
χ_3	1	1	1
χ_{21}	2	0	-1
χ_{111}	1	-1	1

We calculate

$$\langle \chi_{21}^j, \chi_3 \rangle = \langle \chi_{21}^j, \chi_{111} \rangle = \frac{1}{6}(2^j + 2(-1)^j) = \frac{1}{3}(2^{j-1} - (-1)^{j-1}),$$

where $\langle \chi, \chi' \rangle = \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_3} \chi(\sigma) \chi'(\sigma)$ is the usual scalar product. The proposition follows. \square

The same type of computations lead to the following decomposition of the fourth symmetric power of a tensor product.

Proposition 4.3. *We have the following decomposition of $S^4(A \otimes B \otimes C)$ into irreducible $GL(A) \times GL(B) \times GL(C)$ -modules:*

$$\begin{aligned} S^4(A \otimes B \otimes C) = & S_4 S_4 S_4 \oplus S_4 S_{31} S_{31} \oplus S_4 S_{22} S_{22} \oplus S_4 S_{211} S_{211} \oplus S_4 S_{1111} S_{1111} \\ & \oplus S_{31} S_{31} S_{31} \oplus S_{31} S_{31} S_{22} \oplus S_{31} S_{31} S_{211} \oplus S_{31} S_{22} S_{211} \oplus S_{31} S_{211} S_{211} \\ & \oplus S_{31} S_{211} S_{1111} \oplus S_{22} S_{22} S_{22} \oplus S_{22} S_{22} S_{1111} \oplus S_{22} S_{211} S_{211} \oplus S_{211} S_{211} S_{211}. \end{aligned}$$

Here $S_l S_\mu S_\nu$ is to be read as $S_l A \otimes S_\mu B \otimes S_\nu C$ plus permutations giving rise to distinct modules. In particular, $S^4(A \otimes B \otimes C)$ is multiplicity free.

Remark. In $S^5(A \otimes B \otimes C)$ all submodules occurring have multiplicity one except for $S_{311} S_{311} S_{221}$ which has multiplicity two. For higher degrees there is a rapid growth in multiplicities.

Now we try to determine which modules are in the ideals of the secant varieties of $X = \text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k)$. We begin with some simple observations:

Proposition 4.4. *[Inheritance] Suppose that an invariant I of $[l_1] \otimes \cdots \otimes [l_k]$ defines a nonzero embedding of I into $S_{l_1} A_1^* \otimes \cdots \otimes S_{l_k} A_k^* \subset S^d(A_1 \otimes \cdots \otimes A_k)^*$.*

Then, for any vector spaces A'_1, \dots, A'_k such that $\dim A'_i \geq \dim A_i$ for all i , the image of the embedding of $(S_{l_1} A'_1)^ \otimes \cdots \otimes (S_{l_k} A'_k)^*$ in $S^d(A'_1 \otimes \cdots \otimes A'_k)^*$ defined by I , is in $I_d(\sigma_r(\text{Seg}(\mathbb{P}A'_1 \times \cdots \times \mathbb{P}A'_k)))$ if and only if the image of the embedding of $S_{l_1} A_1^* \otimes \cdots \otimes S_{l_k} A_k^*$ in $S^d(A_1 \otimes \cdots \otimes A_k)^*$ defined by I is in $I_d(\sigma_r(\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k)))$.*

Proposition 4.5. *Let $X^{(1)} = \text{Seg}(\mathbb{P}A_2 \times \cdots \times \mathbb{P}A_k)$. Then*

$$I_d(\sigma_{d-1}(X)) \cap (S^d A_1^* \otimes S^d(A_2^* \otimes \cdots \otimes A_k^*)) = S^d A_1^* \otimes I_d(\sigma_{d-1}(X^{(1)})).$$

Say $S_{\pi_1} A_1 \otimes \cdots \otimes S_{\pi_k} A_k \in S^d(A_1 \otimes \cdots \otimes A_k)$. An easy way to verify if it is in $I_d(\sigma_r(X))$ is if corollary 3.4 (3) applies. This method is possible in low degrees but in higher degrees multiplicities appear and the method becomes impossible to use. Thus one either needs to understand the maps (1) or to write down explicit polynomials and test them on $\sigma_r(X)$. One can either test a special polynomial in a module on a general point or test a general polynomial in a module at a special point. The routines we used were more adapted to the first method. We now describe two ways to explicitly write down polynomials. The first has the advantage of producing the entire module, the second of being quicker in producing a polynomial that is a highest weight vector.

Fix π_1, \dots, π_k partitions of d . Compute $\dim([\pi_1] \otimes \cdots \otimes [\pi_k])^{\mathfrak{S}_d}$, call this number m .

ALGORITHM 1

- Explicitly realize the representations $[\pi_j]$ of \mathfrak{S}_d .
- Take independent elements $e_j \in [\pi_j]$ and average $e_1 \otimes \cdots \otimes e_k$ over \mathfrak{S}_d . The result is either a nontrivial invariant I or zero. Continue finding such elements I until one has m independent such.
- Choose embeddings $S_{\pi_j} A_j \rightarrow A_j^{\otimes d}$, the images of the invariants I_s , $1 \leq s \leq m$ give the modules.

Example 1. Let $k = 4$ and $d = 3$. The space of invariants $([21] \otimes [21] \otimes [21] \otimes [21])^{\mathfrak{S}_3}$ has dimension 2. The representation $[21]$ of \mathfrak{S}_3 can be realized as the hyperplane $x_1 + x_2 + x_3 = 0$ in \mathbb{C}_3 , and the action of \mathfrak{S}_3 is to permute the coordinates (x_1, x_2, x_3) . A basis of $[21]$ is given by $e = (1, -1, 0)$ and $f = (0, 1, -1)$. To obtain a basis of the invariants of $[21] \otimes [21] \otimes [21] \otimes [21]$ we consider the natural basis of $[21]^{\otimes 4}$ and apply the averaging operator over all translates by \mathfrak{S}_3 . Applying this procedure to $eeee = e \otimes e \otimes e \otimes e$ and $eeff = e \otimes e \otimes f \otimes f$, we obtain the two invariants

$$\begin{aligned} I_1 &= eeee + (e+f)(e+f)(e+f)(e+f) + ffff, \\ I_2 &= 2eeee + eee f + eef e + e f e e + f e e e + 3eeff + \\ &\quad + 3ffee + fffe + ffef + f e f f + e f f f + 2ffff. \end{aligned}$$

Now consider the space of \mathfrak{S}_3 -equivariant morphisms u from $[21]$ to $V^{\otimes 3}$, where V is any vector space. Let $E = u(e)$. Let s_1 denote the transposition (12) and s_2 the transposition (23). Since $s_1(e) = -e$, we get $s_1(E) = -E$. Since $f = s_2(e) - e$, we have $u(f) = s_2(E) - E$. And since $s_1(f) = e + f$, we must have $E - s_2(E) + s_1 s_2(E) = 0$. The conclusion is that $S_{21}V = \text{Hom}_{\mathfrak{S}_3}([21], V^{\otimes 3})$ is isomorphic to the space of tensors $E \in V^{\otimes 3}$ such that $s_1(E) = -E$ and $E - s_2(E) + s_1 s_2(E) = 0$.

Choose an invariant $J \in ([21] \otimes [21] \otimes [21] \otimes [21])^{\mathfrak{S}_3}$ and consider the embedding of

$$S_{21}A_1 \otimes S_{21}A_2 \otimes S_{21}A_3 \otimes S_{21}A_4$$

in $S^3(A_1 \otimes A_2 \otimes A_3 \otimes A_4)$ that it defines. If $J = \alpha eeee + \cdots + \beta ffff$ and $u_i \in S_{21}A_i$, the corresponding polynomial is defined by the equation

$$\begin{aligned} P_{u_1, u_2, u_3, u_4}^J(a_1 b_1 c_1 d_1, a_2 b_2 c_2 d_2, a_3 b_3 c_3 d_3) = \\ \alpha u_1(e)(a_1 a_2 a_3) u_2(e)(b_1 b_2 b_3) u_3(e)(c_1 c_2 c_3) u_4(e)(d_1 d_2 d_3) + \cdots + \\ + \beta u_1(f)(a_1 a_2 a_3) u_2(f)(b_1 b_2 b_3) u_3(f)(c_1 c_2 c_3) u_4(f)(d_1 d_2 d_3). \end{aligned}$$

Now we evaluate P_{u_1, u_2, u_3, u_4}^J on $\sigma_2(X)$, which means that we let $a_2 = a_1$, $b_2 = b_1$, $c_2 = c_1$, $d_2 = d_1$. Since $u_i(e)$ is skew-symmetric in its first two arguments, its contribution will always

be zero and we get

$$P_{u_1, u_2, u_3, u_4}^J(a_1 b_1 c_1 d_1, a_2 b_2 c_2 d_2, a_3 b_3 c_3 d_3) = \beta u_1(f)(a_1 a_1 a_3) u_2(f)(b_1 b_1 b_3) u_3(f)(c_1 c_1 c_3) u_4(f)(d_1 d_1 d_3),$$

so that the module defined by J is in $I_3(\sigma_2(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_4))$ if and only if $\beta = 0$.

An immediate generalization of this argument leads to the following result:

Proposition 4.6. *The space of modules in $I_3(\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k))$ induced from $[21]^{\otimes k}$ is a codimension one subspace of the modules in $S^3 V^*$ induced from $[21]^{\otimes k}$.*

This proposition allows one to determine the space of cubics vanishing on $\sigma_2(X)$. Indeed, every component of $S^3(A_1 \otimes \cdots \otimes A_k)^*$ involving a wedge power will do. Those involving a symmetric power are determined inductively by Proposition 4.5. The only remaining term is $S_{21}A_1 \otimes \cdots \otimes S_{21}A_k$, whose multiplicity equals $(2^{j-1} - (-1)^{j-1})/3$. The previous proposition means that the subspace vanishing on $\sigma_2(X)$ has multiplicity one less.

Theorem 4.7. *The space of cubics vanishing on the secant variety $\sigma_2(\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k))$ is*

$$\begin{aligned} I_3(\sigma_2(\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k))) = & \bigoplus_{\substack{|I|+|J|+|L|=k, \\ |J|>1, |L|>0}} \frac{2^{j-1} - (-1)^{j-1}}{3} S_3 A_I \otimes S_{21} A_J \otimes S_{111} A_L \\ & \oplus \bigoplus_{\substack{|I|+|J|=k, \\ |J|>1}} \left(\frac{2^{j-1} - (-1)^{j-1}}{3} - 1 \right) S_3 A_I \otimes S_{21} A_J \oplus \bigoplus_{\substack{|I|+|L|=k, \\ |L|>0 \text{ even}}} S_3 A_I \otimes S_{111} A_L. \end{aligned}$$

Corollary 4.8. *Let $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. Then*

$$\begin{aligned} I_3(\sigma_2(X)) = & (S^3 A \otimes \Lambda^3 B \otimes \Lambda^3 C)^* \oplus (\Lambda^3 A \otimes S^3 B \otimes \Lambda^3 C)^* \oplus (\Lambda^3 A \otimes \Lambda^3 B \otimes S^3 C)^* \\ & \oplus (S_{21} A \otimes S_{21} B \otimes \Lambda^3 C)^* \oplus (S_{21} A \otimes \Lambda^3 B \otimes S_{21} C)^* \oplus (\Lambda^3 A \otimes S_{21} B \otimes S_{21} C)^*, \end{aligned}$$

the space of 3×3 minors of the three possible flattenings of $A \otimes B \otimes C$. In particular, letting $a = \dim A$, $b = \dim B$, $c = \dim C$, we have

$$\begin{aligned} \dim I_3(\sigma_2(X)) = & \frac{abc}{72} \left(-6(ab + ac + bc) - 8(a + b + c) + 16 + 27abc - 5(a^2 b^2 c + a^2 b c^2 + a b^2 c^2) \right. \\ & - 3(a^2 b c + a b^2 c + a b c^2) + 5a^2 b^2 c^2 + 2(a^2 b + a^2 c + a b^2 + a c^2 + b^2 c + b c^2) \\ & \left. + 2(a^2 b^2 + 2a^2 c^2 + 2b^2 c^2) \right). \end{aligned}$$

In particular, we recover the data collected in [6] and computed by MACAULAY for the triple Segre products.

Corollary 4.9. *Let A, B, C, D have dimensions a, b, c, d respectively, we get the following number of cubic equations:*

$$\begin{aligned} \dim(I_3(\sigma_2(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C \times \mathbb{P}D))) = \\ \frac{abcd}{1296} \Big(368 - 72(a+b+c+d+ab+ac+ad+bc+bd+cd) - 8(a^2+b^2+c^2+d^2) \\ - 54(abc+abd+acd+bcd) + 8(a^2b^2+a^2c^2+b^2c^2+a^2d^2+b^2d^2+c^2d^2) + 567abcd \\ + 18(a^2bc+ab^2c+abc^2+a^2bd+ab^2d+a^2cd+b^2cd+ac^2d+bc^2d+abd^2+acd^2+bcd^2) \\ - 27abcd(a+b+c+d) + 18(a^2b^2c+a^2bc^2+ab^2c^2+b^2c^2d+a^2b^2d+a^2c^2d+b^2cd^2+ \\ + a^2bd^2+ab^2d^2+a^2cd^2+ac^2d^2+bc^2d^2) + 10(a^2b^2c^2+b^2c^2d^2+a^2b^2d^2+a^2c^2d^2) \\ - 45abcd(cd+bd+ad+bc+ac+ab) - 63abcd(abc+abd+adc+bcd) + 143a^2b^2c^2d^2 \Big). \end{aligned}$$

This recovers all the computations of cubic equations in [6].

Before describing our second algorithm we do some preparation:

Fix a partition $\pi = (p_1, \dots, p_f)$ of size $d = p_1 + \dots + p_f$. For $\alpha_1, \dots, \alpha_f \in A^*$, let

$$F_A = (\alpha_1)^{\otimes (p_1-p_2)} \otimes (\alpha_1 \wedge \alpha_2)^{\otimes (p_2-p_3)} \otimes \dots \otimes (\alpha_1 \wedge \dots \wedge \alpha_{f-1})^{\otimes p_{f-1}-p_f} \otimes (\alpha_1 \wedge \dots \wedge \alpha_f)^{\otimes p_f} \in (A^*)^{\otimes d}.$$

When the $\alpha_1, \dots, \alpha_f$ vary, the subspace of $(A^*)^{\otimes d}$ generated by the F_A 's is a copy of $S_\pi A^*$. In other words, we have defined an element of $\text{Hom}_{GL(A)}(S_\pi A^*, (A^*)^{\otimes d})$, which is isomorphic to $[\pi]$ by Schur duality.

ALGORITHM 2

- For each A_j , choose a basis $\alpha_1^j, \dots, \alpha_{\dim A_j}^j$, and it is better to choose a weight basis with α_1^j a highest weight vector. Continue the notation $m = \dim([\pi_1] \otimes \dots \otimes [\pi_k])^{\mathfrak{S}_d}$.
- Fix elements $\tau_1, \dots, \tau_k \in \mathfrak{S}_d$. Let

$$F(a_1^1 \otimes a_1^2 \otimes \dots \otimes a_1^k, \dots, a_d^1 \otimes a_d^2 \otimes \dots \otimes a_d^k) = F_{A_1}(a_{\tau_1(1)}^1, \dots, a_{\tau_1(d)}^1) \dots F_{A_k}(a_{\tau_k(1)}^k, \dots, a_{\tau_k(d)}^k).$$

By construction $F \in (S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^*$.

- Now let

$$P = \sum_{\sigma \in \mathfrak{S}_d} F_{A_1}(a_{\sigma\tau_1(1)}^1, \dots, a_{\sigma\tau_1(d)}^1) \dots F_{A_k}(a_{\sigma\tau_k(1)}^k, \dots, a_{\sigma\tau_k(d)}^k).$$

By construction $P \in S^d(A_1 \otimes \dots \otimes A_k)^*$, and $P \in (S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^*$ as it is a sum of terms in $(S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^*$. Either P is zero or it gives a nontrivial element of $(S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^* \subset S^d(A_1 \otimes \dots \otimes A_k)^*$.

Note that since we are choosing highest weight vectors, linear combinations will also be highest weight vectors, thus we have a systematic way to look for polynomials even when multiplicities occur.

In practice we implemented the algorithm in two parts as follows:

Input:

- k , the number of vector spaces;
- d_1, \dots, d_k ; their dimensions;
- d , the degree of the polynomial to be constructed;
- π_1, \dots, π_k , partitions of d

PART ONE: Finding the polynomials.

- (1) Calculate $m = \dim([\pi_1] \otimes \dots \otimes [\pi_k])^{\mathfrak{S}_d}$ via a character calculation as in (2).

- (2) Choose a collection of permutations $T_1 = (\tau_1, \dots, \tau_k)$ with $\tau_j \in \mathfrak{S}_d$ (without loss of generality, take $\tau_1 = Id$). Write out F^{T_1} as in (3) above and then average over \mathfrak{S}_d to obtain a polynomial P^{T_1} as above.
- (3) Test if P^{T_1} is identically zero either by a symbolic calculation or by evaluating it at a randomly chosen point. If it is zero return to step 2.
- (4) Repeat steps 2 and 3 for collections of permutations T_2, \dots, T_m , only when repeating step 3, not only test if the polynomial is nonzero, but also test that it is linearly independent from the polynomials already constructed.

Output: a basis of highest weight vectors for the isotypic submodule of copies of $(S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^*$ inside $S^d(A_1 \otimes \dots \otimes A_k)^*$.

PART TWO: Testing if any modules are in the ideal.

Input:

- The polynomials P^{T_1}, \dots, P^{T_m} constructed in part one.
 - p : where we will test for generators of $I_d(\sigma_p(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k))$.
- (1) Write $d = up + r$, with u, r nonnegative integers and $r < p$. Let $P = c_1 P^{T_1} + \dots + c_m P^{T_m}$ where the c_j 's are variables. Pick p vectors in each A_i at random, a_1^i, \dots, a_p^i . Considering P as a multi-linear form, let $\bar{a}_j = a_j^1 \otimes \dots \otimes a_j^k$ evaluate

$$P(\bar{a}_1, \dots, \bar{a}_p, \bar{a}_1, \dots, \bar{a}_p, \dots, \bar{a}_1, \dots, \bar{a}_p, \dots, \bar{a}_1, \dots, \bar{a}_r).$$

- (2) Now pick $m - 1$ more such sets of vectors and solve for the c_j 's.
- (3) For simplicity, say there is a unique solution, test on one more set of vectors using P with the c_j 's replaced by their solution values. If one gets zero, one has a good candidate.

Warning: this is just one of many tests to perform to see if a candidate is in the ideal - we begin with this one only because in practice it has been quite useful. Hence the next step:

- (4) Now test P on all possible ways of choosing the last r vectors from the set of first p vectors (e.g., one needs to test the possibility of the first vector occurring r times instead of r different vectors etc...). Ideally do this symbolically, but one gets an answer with very high probability by testing at random points.

Output: Either ruling out the modules $(S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^* \subset S^d(A_1 \otimes \dots \otimes A_k)^*$ from being in $I_d(\sigma_p(X))$ or determination of an explicit copy of $(S_{\pi_1} A_1 \otimes \dots \otimes S_{\pi_k} A_k)^* \subset S^d(A_1 \otimes \dots \otimes A_k)^*$ that is in $I_d(\sigma_p(X))$ described by its highest weight vector.

Here are some examples:

Example 2. Consider $S_{211}A \otimes S_{211}B \otimes S_{211}C \subset S^4(A \otimes B \otimes C)$. Without loss of generality assume $\dim A = \dim B = \dim C = 3$. Take

$$F(a_1 b_1 c_1, \dots, a_4 b_4 c_4) = \alpha^1(a_1) \det(a_2, a_3, a_4) \beta^1(b_2) \det(b_1, b_3, b_4) \gamma^1(c_3) \det(c_1, c_2, c_4)$$

and let P be the corresponding polynomial. A simple evaluation at a random point shows P is not identically zero. (Compare with taking all permutations τ to be the identity, then the average over \mathfrak{S}_4 is indeed zero.)

Example 3. Consider $S_{333}A \otimes S_{333}B \otimes S_{333}C$. Without loss of generality take $\dim A = \dim B = \dim C = 3$. We take

$$F = \det(a_1, a_2, a_3) \det(a_4, a_5, a_6) \det(a_7, a_8, a_9) \det(b_2, b_3, b_4) \det(b_5, b_6, b_7) \\ \det(b_1, b_8, b_9) \det(c_3, c_4, c_5) \det(c_6, c_7, c_8) \det(c_1, c_2, c_9)$$

Here it is more delicate to see the corresponding polynomial P is not identically zero because there will be terms that appear several times. One needs to check that they do not have signs cancelling. (For example, had we had any pair of indices occuring three times in a determinant, the corresponding polynomial would be zero because the transposition of the indices would produce the same terms with opposite signs.) One can also verify with Maple that the corresponding polynomial is nonzero.

Example 4. Consider $S_{321}A \otimes S_{321}B \otimes S_{3111}C \subset S^6(A \otimes B \otimes C)$ which occurs with multiplicity four. Let

$$\begin{aligned} F_{\tau,\mu} = & \alpha_1 \wedge \alpha_2 \wedge \alpha_3(a_1, a_2, a_3) * \alpha_1 \wedge \alpha_2(a_5, a_6) * \alpha_1(a_4) \\ & \beta_1 \wedge \beta_2 \wedge \beta_3(b_{\tau(1)}, b_{\tau(2)}, b_{\tau(3)}) * \beta_1 \wedge \beta_2(b_{\tau(4)}, b_{\tau(5)}) * \beta_1(b_{\tau(6)}) \\ & \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4(c_{\mu(1)}, c_{\mu(2)}, c_{\mu(3)}, c_{\mu(4)}) * \gamma_1(c_{\mu(5)}) * \gamma_1(c_{\mu(6)}) \end{aligned}$$

Now we take the following permutations:

$$\begin{aligned} \tau_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 2 & 6 \end{pmatrix} & \mu_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 2 & 3 \end{pmatrix} \\ \tau_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 2 & 6 \end{pmatrix} & \mu_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix} \\ \tau_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 2 & 6 \end{pmatrix} & \mu_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 & 6 \end{pmatrix} \\ \tau_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 1 & 2 & 5 \end{pmatrix} & \mu_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 & 6 \end{pmatrix} \end{aligned}$$

The resulting four polynomials, call them P_1, \dots, P_4 are linearly independent. We verified this by evaluating them first at four random points to determine a unique possible linear combination that is zero, and then evaluated this linear combination at a fifth random point - one does not obtain zero.

Remark. When the A_i 's have the same dimension k , $S^k(A_1 \otimes \dots \otimes A_k)^*$ contains a copy of $\mathfrak{sl}(A_1) \otimes \dots \otimes \mathfrak{sl}(A_k)$, with an embedding given by the formula

$$P_{X_1, \dots, X_k}(a_1^1 \otimes \dots \otimes a_k^1, \dots, a_1^k \otimes \dots \otimes a_k^k) = (X_1 a_1^1 \wedge a_1^2 \wedge \dots \wedge a_1^k) \dots (X_k a_k^1 \wedge a_k^2 \wedge \dots \wedge a_k^k) + \text{symmetric terms.}$$

All such polynomials vanish on $\sigma_{k-3}(X)$, but not on $\sigma_{k-2}(X)$.

5. FLATTENINGS AND THE GSS CONJECTURE

Let $X = \text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k) \subset \mathbb{P}(A_1 \otimes \dots \otimes A_k)$. A family of degree $d+1$ equations for $\sigma_d(X)$ is given by the *flattenings*, discussed in [6].

Definition 5. Given $V = A_1 \otimes \dots \otimes A_k$, a *flattening* of V is a decomposition

$$V = (A_{i_1} \otimes \dots \otimes A_{i_q}) \otimes (A_{j_1} \otimes \dots \otimes A_{j_{k-q}}) = A_I \otimes A_J$$

where $I + J = \{1, \dots, k\}$ is a partition of $\{1, \dots, k\}$ into two subsets.

Since $X \subset \text{Seg}(\mathbb{P}A_I \times \mathbb{P}A_J)$, $\sigma_k(X) \subseteq \sigma_k(\text{Seg}(\mathbb{P}A_I \times \mathbb{P}A_J))$ and thus the $(d+1) \times (d+1)$ minors of flattenings always vanish on $\sigma_d(X)$, i.e.

$$\wedge^{d+1}(A_{i_1} \otimes \dots \otimes A_{i_q})^* \otimes \wedge^{d+1}(A_{j_1} \otimes \dots \otimes A_{j_{k-q}})^* \subset I_{d+1}(\sigma_d(X)).$$

In [6] it was conjectured that $I(\sigma_2(X))$ is generated by the 3×3 minors of flattenings, i.e., that $\sigma_2(X)$ is intersection as a scheme of the varieties $\sigma_2(\mathbb{P}A_I \times \mathbb{P}A_J)$. We will prove this for

$k = 3$ below. For $k > 3$ we have the following partial result which implies that $\sigma_2(X)$ is intersection as a set of the varieties $\sigma_2(\mathbb{P}A_I \times \mathbb{P}A_J)$.

Theorem 5.1. *Let $X = \text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_k)$ be a Segre product of projective spaces.*

- *The first secant variety $\sigma_2(X)$ is defined set theoretically by the 3×3 minors of flattenings.*
- *$I_3(\sigma_2(X))$ is spanned by the 3×3 minors of flattenings.*

The corresponding modules were described explicitly in Theorem 4.7.

Proof. Let $T \in A_1 \otimes \cdots \otimes A_k$ be a tensor on which the 3×3 minors of flattenings all vanish. This means that T has rank two at most, when considered as a tensor of $A_I \otimes A_J$, where $A_I = A_{i_1} \otimes \cdots \otimes A_{i_q}$ and $A_J = A_{j_1} \otimes \cdots \otimes A_{j_{k-q}}$, with $V = A_I \otimes A_J$ any flattening. Applying this to the case where $\#I = 1$, we see that we can find two dimensional subsets $A'_i \subset A_i$ such that $T \in A'_1 \otimes \cdots \otimes A'_k$. In other words, we may and will suppose that $\dim A_i = 2$ for all i .

Now take $I = \{1, 2\}$. We can decompose our tensor as $T = M \otimes S + M' \otimes S'$, where $M, M' \in A_1 \otimes A_2$. We can identify A_1 with the dual of A_2 and consider M and M' as endomorphisms of A_2 . Suppose that one of them has rank two. We can adapt our basis so that M , for example, is the identity and M' is in Jordan canonical form. Generically, M' will be diagonalizable and we can rewrite our tensor as

$$T = a_1 \otimes a_2 \otimes C + a'_1 \otimes a'_2 \otimes C'.$$

If a_1 and a'_1 , or a_2 and a'_2 , are proportional, T can be factored as $a_1 \otimes U$ and we are reduced to the case of $k - 1$ factors. So we can suppose that (a_1, a'_1) is a basis of A_1 , and (a_2, a'_2) a basis of A_2 . Then we consider C and C' as map from $(A_3 \otimes \cdots \otimes A_k)^*$ to A_3 and apply our hypothesis to the set of indices $I = \{1, 3\}$. The conclusion is that $a_1 \otimes C(t)$ and $a'_1 \otimes C'(t)$ belong to a fixed two-dimensional subset of $A_1 \otimes A_3$, as t varies in $(A_3 \otimes \cdots \otimes A_k)^*$. Since a_1 and a'_1 are independant, this implies that C and C' have rank one. But the same conclusion holds if we replace $I = \{1, 3\}$ by any $I = \{1, j\}$, $j \geq 3$, and this means that we can decompose $C = a_3 \otimes \cdots \otimes a_k$ and $C' = a'_3 \otimes \cdots \otimes a'_k$. Thus T belongs to the secant variety $\sigma_2(X)$.

Suppose now that M' is not diagonalizable. Then we can find bases (a_1, a'_1) of A_1 , and (a_2, a'_2) of A_2 , such that we can decompose T as

$$T = (a'_1 \otimes a_2 + a_1 \otimes a'_2) \otimes C + a_1 \otimes a_2 \otimes C'.$$

We shall prove by induction on $j \geq 2$ that we can decompose T further as

$$T = (a'_1 \otimes a_2 \otimes \cdots \otimes a_j + \cdots + a_1 \otimes \cdots \otimes a_{j-1} \otimes a'_j) \otimes C_j + a_1 \otimes \cdots \otimes a_j \otimes C'_j,$$

for some $C_j, C'_j \in A_{j+1} \otimes \cdots \otimes A_k$. As in the previous case, we consider C_j and C'_j as morphisms from $(A_{j+2} \otimes \cdots \otimes A_k)^*$ to A_{j+1} . Then $a_2 \otimes \cdots \otimes a_j \otimes C_j(t)$ and $(a'_2 \otimes \cdots \otimes a_j + \cdots + a_2 \otimes \cdots \otimes a'_j) \otimes C_j(t) + a_2 \otimes \cdots \otimes a_j \otimes C'_j(t)$ belong to a fixed two dimensional space V_j as t varies. This implies that C_j has rank one, we write it as $C_j = a_{j+1} \otimes C_{j+1}$. Then V_j contains the tensors $a_2 \otimes \cdots \otimes a_{j+1}$, $(a'_2 \otimes \cdots \otimes a_j + \cdots + a_2 \otimes \cdots \otimes a'_j) \otimes a_{j+1} + a_2 \otimes \cdots \otimes a_j \otimes C'_j(t_0)$ for $C_{j+1}(t_0) = 1$, and $a_2 \otimes \cdots \otimes a_j \otimes C'_j(t)$ for t in the kernel of C . But the first two vectors are already independant, so that those of the third type must be proportional to the first one, which means that C'_j maps the kernel of C_j its image. But this means that we can decompose $C'_j = a_{j+1} \otimes C'_{j+1} + a'_{j+1} \otimes C_{j+1}$ for some $a'_{j+1} \in A_{j+1}$ and $C_{j+1} \in A_{j+2} \otimes \cdots \otimes A_k$. This concludes the induction.

When $j = k - 1$, we finally get a decomposition of T as

$$T = a'_1 \otimes a_2 \otimes \cdots \otimes a_k + \cdots + a_1 \otimes \cdots \otimes a_{k-1} \otimes a'_k + a_1 \otimes \cdots \otimes a_k.$$

We conclude that T belongs to the (affine) tangent space of X at the point $a_1 \otimes \cdots \otimes a_k$. In particular, T belongs to the tangential variety of X , which is contained in the secant variety $\sigma_2(X)$. \square

Theorem 5.2. *Let $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$ be a triple Segre product. Then the ideal of the secant variety $\sigma_2(X)$ is generated by cubics.*

Proof. Let $\hat{\sigma} \subset A \otimes B \otimes C$ denote the cone over $\sigma_2(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. Let $G_2(A)$ denote the Grassmanian of two-planes in A . Consider the quasiprojective variety $Y = G_2(A) \times G_2(B) \times G_2(C) \times A \otimes B \otimes C$, and denote by p and π its projections to $G_2(A) \times G_2(B) \times G_2(C)$ and $A \otimes B \otimes C$. Let T_A denote the tautological rank two vector bundle on $G_2(A)$, and let E denote the vector bundle $A \otimes B \otimes C / T_A \otimes T_B \otimes T_C$ on $G_2(A) \times G_2(B) \times G_2(C)$. The pull-back p^*E has a canonical section s , defined by

$$s(U_A, U_B, U_C, t) = t \pmod{U_A \otimes U_B \otimes U_C}.$$

Let $\tilde{\sigma}$ denote the zero-locus of this section.

Lemma 5.3. *The zero-locus $\tilde{\sigma}$ is a vector bundle over $G_2(A) \times G_2(B) \times G_2(C)$, in particular it is a smooth variety. Its image under π is $\hat{\sigma}$, and the restriction map $\pi|_{\tilde{\sigma}} : \tilde{\sigma} \rightarrow \hat{\sigma}$ is a resolution of singularities.*

Proof. The first assertion is clear. The second one is an immediate consequence of the fact that the secant variety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ is non degenerate, i.e., equal to \mathbb{P}^7 . \square

Consider the Koszul complex of the section s : this is a minimal free resolution of the structure sheaf of $\tilde{\sigma}$. We want to push it down to $A \otimes B \otimes C$ to get some information on the minimal resolution of $\hat{\sigma}$. For this we use the spectral sequence

$$\mathcal{E}_1^{p,q} = R^q \pi_* p^*(\wedge^{-p} E^*) \implies R^{p+q} \pi_* \mathcal{O}_{\tilde{\sigma}}.$$

Lemma 5.4. *We have $R^q \pi_* \mathcal{O}_{\tilde{\sigma}} = 0$ for $q > 0$ and $\pi_* \mathcal{O}_{\tilde{\sigma}} = \mathcal{O}_{\hat{\sigma}}$. In particular, $\hat{\sigma}$ has rational singularities.*

Proof. The fibers of π are isomorphic to $G_2(A) \times G_2(B) \times G_2(C)$, and the vector bundle p^*E is a pull-back from that product. This reduces the problem to the computation of the cohomology of E^* and its exterior powers. Let V denote the trivial bundle whose fiber is isomorphic to $A \otimes B \otimes C$ and let $T = T_A \otimes T_B \otimes T_C$. For each integer r , we have an exact sequence

$$0 \rightarrow \wedge^r E^* \rightarrow \wedge^r V^* \rightarrow \wedge^{r-1} V^* \otimes T^* \rightarrow \wedge^{r-2} V^* \otimes S^2 T^* \rightarrow \cdots \rightarrow S^r T^* \rightarrow 0.$$

By Bott's theorem, the vector bundles $S^k T^*$ are acyclic. Since the previous resolution of $\wedge^r E^*$ is of length $r+1$, this implies that

$$\begin{aligned} H^q(G_2(A) \times G_2(B) \times G_2(C), \wedge^r E^*) &= 0 \quad \text{for } q > r, \\ H^r(G_2(A) \times G_2(B) \times G_2(C), \wedge^r E^*) &= \text{coker} \left(V^* \otimes H^0(S^{r-1} T^*) \longrightarrow H^0(S^r T^*) \right). \end{aligned}$$

The first claim implies that in the spectral sequence ($r = -p$!), $\mathcal{E}_1^{p,q} = 0$ for $p+q > 0$, hence $R^k \pi_* \mathcal{O}_{\tilde{\sigma}} = 0$ for $k > 0$.

To prove that $\pi_* \mathcal{O}_{\tilde{\sigma}} = \mathcal{O}_{\hat{\sigma}}$, we need to check that $H^r(G_2(A) \times G_2(B) \times G_2(C), \wedge^r E^*) = 0$ for $r > 0$. But note that if

$$\begin{aligned} S^r V^* &= \oplus_{l,\mu,\nu} c_{l,\mu,\nu} S_l A^* \otimes S_\mu B^* \otimes S_\nu C^*, \\ \text{then } S^r T^* &= \oplus_{l(l), l(\mu), l(\nu) \leq 2} c_{l,\mu,\nu} S_l T_A^* \otimes S_\mu T_B^* \otimes S_\nu T_C^*, \\ \text{thus } H^0(S^r T^*) &= \oplus_{l(l), l(\mu), l(\nu) \leq 2} c_{l,\mu,\nu} S_l A^* \otimes S_\mu B^* \otimes S_\nu C^*. \end{aligned}$$

Therefore, if $r > 0$, the surjectivity of the map $V^* \otimes H^0(S^{r-1}T^*) \longrightarrow H^0(S^rT^*)$ is an immediate consequence of the surjectivity of $V^* \otimes S^{r-1}V^* \longrightarrow S^rV^*$. \square

We are now in position to apply Theorem (5.1.3) of [14], following which the vector bundles $R^q\pi_*p^*(\wedge^{-p}E^*)$ can be organized into a resolution of $\mathcal{O}_{\hat{\sigma}}$. In particular, the cohomology groups $H^{r-1}(\wedge^r E^*)$ appear as degree r equations of $\hat{\sigma}$. Thus, if we can prove that these groups vanish for $r \neq 3$, we'll get that the ideal of $\hat{\sigma}$ is generated by cubics.

Using the previous resolution of $\wedge^r E^*$, we see that $H^{r-1}(\wedge^r E^*)$ is the homology group of the complex on the first line of the diagram

$$\begin{array}{ccccc} \wedge^2 V^* \otimes H^0(S^{r-2}T^*) & \longrightarrow & V^* \otimes H^0(S^{r-1}T^*) & \longrightarrow & H^0(S^rT^*) \\ \uparrow & & \uparrow & & \uparrow \\ \wedge^2 V^* \otimes S^{r-2}V^* & \longrightarrow & V^* \otimes S^{r-1}V^* & \longrightarrow & S^rV^* \end{array}$$

The complex on the lowest line is a Koszul complex. It is exact, and surjects onto the complex we are interested in. For $r = 1$ or $r = 2$ we get the same complexes, hence $H^0(E^*) = H^1(\wedge^2 E^*) = 0$. For $r = 3$ only the rightmost terms are different, and $H^0(S^3T^*)$ is the sum of components in S^3V^* without terms of length three. The other components give $H^2(\wedge^3 E^*)$.

Next we must prove that $H^{r-1}(\wedge^r E^*) = 0$ for $r \geq 4$. First observe that the components of $V^* \otimes H^0(S^{r-1}T^*)$ have length three at most. Those of length at most two on each factor map to $H^0(S^rT^*)$ as they do in the Koszul complex, which is exact: this takes care of that kind of terms. Now consider an isotypical component D inside $V^* \otimes H^0(S^{r-1}T^*)$ with length three, say, on A . This component maps to zero in $H^0(S^rT^*)$, and we must check that it belongs to the image of $\wedge^2 V^* \otimes H^0(S^{r-2}T^*)$. But we know that the secant variety $\text{Seg}(\mathbb{P}A \times \mathbb{P}(B \otimes C))$ is cut out by cubics, and this implies that the corresponding complex

$$\wedge^2 V^* \otimes H^0(S^{r-2}U^*) \longrightarrow V^* \otimes H^0(S^{r-1}U^*) \longrightarrow H^0(S^rU^*)$$

is exact. Here U is the tautological vector bundle $T_A \otimes T_{B \otimes C}$ on $G_2(A) \times G_2(B \otimes C)$, so that

$$H^0(S^rU^*) = \oplus_{l(l) \leq 2, \mu, \nu} c_{l, \mu, \nu} S_l A^* \otimes S_\mu B^* \otimes S_\nu C^*.$$

Therefore, we see our component D inside $V^* \otimes H^0(S^{r-1}U^*)$, and for the same reason as before, it maps to zero in $H^0(S^rU^*)$. So it must belong to the image of $\wedge^2 V^* \otimes H^0(S^{r-2}U^*)$. We must check that in fact, it only comes from components of $H^0(S^{r-2}U^*)$ with length at most two on each factor. But this is clear, because the contraction map factors as

$$\wedge^2 V^* \otimes H^0(S^{r-2}U^*) \longrightarrow V^* \otimes (V^* \otimes H^0(S^{r-2}U^*)) \longrightarrow V^* \otimes H^0(S^{r-1}U^*).$$

This implies that a component of $H^0(S^{r-2}U^*)$ with length greater than two on some factor will give components of $V^* \otimes H^0(S^{r-2}U^*)$ with the same property, and these cannot contribute to D . \square

6. EQUATIONS OF $\sigma_k(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$

In this section we analyze the equations of the secant varieties of $\text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$ for low dimensional vector spaces A, B, C .

For the dimensions of the secant varieties and filling k that we use in this subsection, we refer the reader to [3, 11].

6.1. Case of $X = \text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)$.

Recall that here $\dim \sigma_k(X) = k(m+n+2-k) - 1$ until it fills and $I(\sigma_k(X))$ is generated in degree $k+1$ by $\Lambda^{k+1}\mathbb{C}^{m+1} \otimes \Lambda^{k+1}\mathbb{C}^{n+1}$.

6.2. Case of $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

Here $\sigma_2(X) = \mathbb{P}V$ and thus its ideal is zero.

6.3. Case of $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^c)$, $c = 2, 3$.

Here $I(\sigma_2(X))$ is generated in degree three by flattenings and $\sigma_3(X) = \mathbb{P}V$.

6.4. Case of $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2)$.

Again $I(\sigma_2(X))$ is generated in degree three by flattenings and $\sigma_3(X) = \mathbb{P}V$.

6.5. Case of $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$.

Again $I(\sigma_2(X))$ is generated in degree three by flattenings.

Proposition 6.1. *Let $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) = \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$.*

- *The space of quartic equations of $\sigma_3(X)$ is*

$$I_4(\sigma_3(X)) = S_{211}A \otimes S_{211}B \otimes S_{211}C,$$

and has dimension 27.

- *The hypersurface $\sigma_4(X)$ is of degree nine and corresponds to the one-dimensional module $S_{333}A \otimes S_{333}B \otimes S_{333}C$.*

A determinantal representation of these equations was given by Strassen, see [6]. We don't know if $I(\sigma_3(X))$ is generated in degree four.

This case is discussed in [6] (without proofs). To study $I_4(\sigma_3(X))$ we need only look at terms $S_{l_1}A \otimes S_{l_2}B \otimes S_{l_3}C$ with each l_j of length 3 by case 6.4 since otherwise, by inheritance (proposition 4.4), we would have a nonzero element in $I_4(\sigma_3(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2))$. Examining the decomposition of $S^4(A \otimes B \otimes C)$ the only possible term is $W = S_{211}A \otimes S_{211}B \otimes S_{211}C$, which occurs with multiplicity one.

To illustrate our methods, we give three proofs that $I_4(\sigma_3(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))) = W$.

First proof: We apply Proposition 3.4, that is, we check that W is not contained in $V^2 \otimes S^2(V)$. This is easy: each term in S^2V must have at least one symmetric power, say S^2A . If we tensor by the other S^2A coming from V^2 , we do not get the $S_{211}A$ term of W .

Second proof: We make explicit the embedding of W^* in $S^4(A \otimes B \otimes C)$, using the first algorithm explained in §4. The representation [211] of \mathfrak{S}_4 is the tensor product of the natural three dimensional representation [31], given by the natural action of \mathfrak{S}_4 on the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ in \mathbb{C}^4 , with the sign representation. We choose the basis $e = (1, -1, 0, 0)$, $f = (0, 1, -1, 0)$, $g = (0, 0, 1, -1)$. We keep this basis for [211] but recall that the action of the symmetric group must be twisted by the sign. Averaging $e \otimes e \otimes f$ over the symmetric group, we obtain a non zero invariant I in $[211] \otimes [211] \otimes [211]$,

$$\begin{aligned} I = & e \otimes e \otimes (-e + f + g) - f \otimes f \otimes (e - f + g) + g \otimes g \otimes (e + f - g) \\ & + (f - e) \otimes (f - e) \otimes (-e - f + g) + (g - e) \otimes (g - e) \otimes (e - f + g) \\ & + (g - f) \otimes (g - f) \otimes (e - f - g). \end{aligned}$$

Now we need to evaluate the corresponding space of polynomials on $\sigma_3(X)$, which means that we may suppose, for example, that the third and fourth arguments are equal decomposed tensors. Let s_3 denote the simple transposition (34). Since $s_3(e) = -e$, the contribution of all terms involving e will vanish. Moreover, $g = -s_3(f)$, and since the contributions of f and $s_3(f)$ in our evaluation are obviously the same, f and $-g$ have the same contribution. But if we let $e = 0$ and $g = f$ in the expression of the invariant I , we get zero, which means that our evaluation on $\sigma_3(X)$ does vanish.

Third proof: We use the second algorithm of section §4. Taking the polynomial P of example 2 we see that if any two vectors are equal, each term of P vanishes.

We now study $\sigma_4(X)$. Since it is a hypersurface by [3, 11], we need to look for instances of the trivial representation in $S^d(A \otimes B \otimes C)$. The first candidate appears when $d = 6$, since

$S^6(A \otimes B \otimes C)$ contains $S_{222}A \otimes S_{222}B \otimes S_{222}C$. The second candidate appears when $d = 9$, since $S^9(A \otimes B \otimes C)$ contains $S_{333}A \otimes S_{333}B \otimes S_{333}C$, again with multiplicity one. It is claimed in [6] that the degree nine equation is the equation of $\sigma_4(X)$. We verify this by applying Proposition 3.4. We need to see if the one dimensional representation $S_{333}A \otimes S_{333}B \otimes S_{333}C$ occurs in either $S^2V^3 \otimes V^2 \otimes V$ or $V^3 \otimes S^3V^2$. In the first factor of the first module, at least one of A, B, C , say A , must occur as $S^2(S^3A) = S_6A \otimes S_{42}A$. Since the partitions (6) and (42) are not contained in (333), we cannot get $S_{333}A$ after tensoring by $V^2 \otimes V$. For the second module, we note that $S_{333}A \otimes S_{333}B \otimes S_{333}C$ could only come from a factor $S_{33}A \otimes S_{33}B \otimes S_{33}C$ of S^3V^2 . But $S^3(P \otimes Q \otimes R)$ does not contain $S_{111}P \otimes S_{111}Q \otimes S_{111}R$, so that up to symmetry, either $S^3(S^2A)$ or $S_{21}(S^2A)$ must occur in each factor, and none of these contains $S_{33}A$.

Now that we have a nonzero polynomial of degree nine that vanishes on the invariant hypersurface $\sigma_4(X)$, we conclude that it must be the equation of this hypersurface. Indeed, suppose not. Then our polynomial would be the product of two polynomials, which automatically would be both invariant. In particular, their degrees would be multiples of three, so one of them would have degree three. But there is no invariant cubic polynomial.

The polynomial is described explicitly in example 3 and one can verify that it does indeed vanish on $\sigma_4(X)$, but some care must be taken in keeping track of the signs. Similarly, one can explicitly write out the polynomial in $S_{222}A \otimes S_{222}B \otimes S_{222}C$ and see that it does not vanish on $\sigma_4(X)$ (for example, this is easy to verify with Maple).

6.6. Case of $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3)$.

Proposition 6.2. *Let $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3) = \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. Then*

$$I_4(\sigma_3(X)) = S_{31}A \otimes S_{211}B \otimes S_{1111}C \oplus S_{22}A \otimes S_{22}B \otimes S_{1111}C$$

is of dimension 6 and generates $I(\sigma_3(X))$.

Also, $\sigma_4(X) = \mathbb{P}V$.

Proof. By [3, 11], $\dim \sigma_3(X) = 20$. On the other hand, $X \subseteq \text{Seg}(\mathbb{P}^5 \times \mathbb{P}^3)$ and $\dim \sigma_3(\text{Seg}(\mathbb{P}^5 \times \mathbb{P}^3)) = 20$ (see case 6.1 above). Since both are irreducible varieties, they are equal and $I_4(\sigma_3(X)) = \Lambda^4(A \otimes B) \otimes \Lambda^4C$. \square

6.7. Case of $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.

Proposition 6.3. *Let $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3) = \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. Then*

- *The space of quartics on X is*

$$I_4(\sigma_3(X)) = S_{211}A \otimes S_{211}B \otimes S_{211}C \oplus S_{31}A \otimes S_{211}B \otimes S_{1111}C \\ \oplus S_{211}A \otimes S_{31}B \otimes S_{1111}C \oplus S_{22}A \otimes S_{22}B \otimes S_{1111}C$$

and is of dimension $135 + 2 \times 45 + 36 = 261$.

- $I_5(\sigma_4(X)) = 0$.
- $I_6(\sigma_4(X)) = S_{222}A \otimes S_{222}B \otimes S_{3111}C \oplus S_{321}A \otimes S_{321}B \otimes S_{3111}C$
The second term occurs with multiplicity one in I_6 while occurring with multiplicity four in S^6V . Note that $\dim I_6(\sigma_4(X)) = 260$ and it does not generate $I(\sigma_4(X))$ because the inherited $S_{333}A \otimes S_{333}B \otimes S_{333}C$ term in $I_9(\sigma_4(X))$ cannot come from these terms.
- $\sigma_5(X) = \mathbb{P}V$

We do not know if $I(\sigma_3(X))$ is generated in degree four.

To determine $I_4(\sigma_3(X))$, in addition to $\Lambda^4(A \otimes B) \otimes \Lambda^4C$, we inherit $S_{211}A \otimes S_{211}B \otimes S_{211}C$ from case 6.5. No other terms are possible by the same argument as in case 6.5.

To see $I_5(\sigma_4(X))$ is empty we explicitly wrote down highest weight vectors in all the possible modules and tested them at random points of $\sigma_4(X)$ with Maple. We used the same method to find the modules in and not in $I_6(\sigma_5(X))$, but when we found a polynomial that vanished, we checked the result symbolically.

In example 4 we gave an explicit basis of the highest weight vectors for $S_{321}A \otimes S_{321}B \otimes S_{3111}C$. The linear combination that vanishes on $\sigma_4(X)$ is the polynomial obtained by symmetrizing $6F_1 - F_2 - 4F_3 + 5F_4$.

The last assertion is not in [3, 11] so we present a proof:

Proof. We use Terracini's lemma. Let $e_1, \dots, e_3, f_1, \dots, f_3, g_1, \dots, g_4$ respectively denote bases of $\mathbb{C}^3, \mathbb{C}^3, \mathbb{C}^4$. For our five points on X , take $e_1 \otimes f_1 \otimes g_1, e_2 \otimes f_2 \otimes g_2, e_3 \otimes f_3 \otimes g_3, (e_1 + e_2 + e_3) \otimes (f_1 + f_2 + f_3) \otimes g_4, (e_1 + e_2 + e_3) \otimes (f_1 + \alpha f_2 + \beta f_3) \otimes (g_1 + g_2 + g_3 + g_4)$ where α, β are relatively prime and $|\alpha - \beta| \neq 1$. An easy calculation shows that if we use the monomial basis except for using $e_1 \otimes f_2 \otimes g_3 + e_2 \otimes f_1 \otimes g_3, e_2 \otimes f_3 \otimes g_1 + e_3 \otimes f_2 \otimes g_1, e_1 \otimes f_3 \otimes g_2 + e_3 \otimes f_1 \otimes g_2, e_1 \otimes f_3 \otimes g_4 + e_3 \otimes f_1 \otimes g_4, e_1 \otimes f_2 \otimes g_3 - e_2 \otimes f_1 \otimes g_3, e_2 \otimes f_3 \otimes g_1 - e_3 \otimes f_2 \otimes g_1, e_1 \otimes f_3 \otimes g_2 - e_3 \otimes f_1 \otimes g_2, e_1 \otimes f_3 \otimes g_4 - e_3 \otimes f_1 \otimes g_4$, instead of the monomials that appear in them, then the span of the tangent spaces to the first four points is all but the last four terms, and adding tangent space to the fifth point enables us to dispense with those. (Recall that $\hat{T}_{[e \otimes f \otimes g]} \text{Seg}(\mathbb{P}E \times \mathbb{P}F \times \mathbb{P}G) = E \otimes f \otimes g + e \otimes F \otimes g + e \otimes f \otimes G$.) \square

6.8. Case of $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$.

Proposition 6.4. *Let $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3) = \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. Then*

- *The space of quartics on $\sigma_3(X)$ is*

$$\begin{aligned} I_4(\sigma_3(X)) = & S_{211}A \otimes S_{31}B \otimes S_{1111}C \oplus S_{211}A \otimes S_{1111}B \otimes S_{31}C \\ & \oplus S_{22}A \otimes S_{22}B \otimes S_{1111}C \oplus S_{22}A \otimes S_{1111}B \otimes S_{22}C \\ & \oplus S_{31}A \otimes S_{1111}B \otimes S_{211}C \oplus S_{31}A \otimes S_{211}B \otimes S_{1111}C \\ & \oplus S_{4}A \otimes S_{1111}B \otimes S_{1111}C \oplus S_{211}A \otimes S_{211}B \otimes S_{211}C \end{aligned}$$

and has dimension $2 \times 135 + 2 \times 120 + 2 \times 225 + 15 + 675 = 1650$.

- *The space of quintic equations of $\sigma_4(X)$ is*

$$I_5(\sigma_4(X)) = S_{311}A \otimes S_{2111}B \otimes S_{2111}C,$$

and has dimension 96. $I(\sigma_4(X))$ is not generated in degree five.

- $I_6(\sigma_5(X)) = 0$.
- $I_7(\sigma_5(X)) = 0$.

Note that $\sigma_6(X)$ fills, see [3, 11].

We do not know if the ideal of $\sigma_3(X)$ is generated in degree four.

Proof. $I_4(\sigma_3(X))$ follows from flattenings and inheritance.

Continuing to $\sigma_4(X)$, since $I_5(\sigma_4(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))) = 0$, the only possible term in $I_5(\sigma_4(X))$ is $W = S_{311}A \otimes S_{2111}B \otimes S_{2111}C$, which occurs in $S^5(A \otimes B \otimes C)$ with multiplicity one, because this is the unique component with a partition of length three in the A factor and length four in the B, C factors. But this factor does not occur inside $S^3V \otimes V^2$. Thus Proposition 3.4 applies.

Turning to $\sigma_5(X)$, since σ_5 fills for both $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3$ we only need look at elements of $S^6(A \otimes B \otimes C)$ that are of length three in the first factor and four in the second and third factors. Examining the decomposition, the candidate modules (up to permutation in the last two factors) are: $S_{411}A \otimes S_{3111}B \otimes S_{3111}C, S_{411}A \otimes S_{2211}B \otimes S_{2211}C, S_{222}A \otimes S_{3111}B \otimes S_{3111}C, S_{222}A \otimes S_{2211}B \otimes S_{2211}C$ with multiplicity one and $S_{411}A \otimes S_{3111}B \otimes S_{2211}C, S_{321}A \otimes S_{3111}B \otimes S_{3111}C, S_{321}A \otimes S_{3111}B \otimes S_{2211}C, S_{321}A \otimes S_{2211}B \otimes S_{2211}C$ with multiplicity two.

On the other hand, consider $S^4V \otimes V^2$. In order to have two modules with partition of length four, we need the partitions in S^4V to have length at least three. All are accounted for, so Proposition 3.4 is not useful here. Thus we do direct calculations with Maple, which is what we use for $I_7(\sigma_5(X))$ as well, the latter being quite involved as modules appear with multiplicity up to nine. \square

6.9. Case of $X = \text{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$.

Proposition 6.5. *Let $X = \text{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3) = \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. Then*

- $I_4(\sigma_3(X)) = \Lambda^4(A \otimes B) \otimes \Lambda^4C$ plus permutations and $S_{211}A \otimes S_{211}B \otimes S_{211}C$.
- The space of quintic equations of $\sigma_4(X)$ is

$$I_5(\sigma_4(X)) = S_{311}A \otimes S_{2111}B \otimes S_{2111}C \oplus S_{2111}A \otimes S_{311}B \otimes S_{2111}C \oplus S_{2111}A \otimes S_{2111}B \otimes S_{311}C$$

and has dimension $3 \times 36 \times 4 \times 4 = 1728$.

We also know that $S_{333}A \otimes S_{333}B \otimes S_{333}C$ is in $I_9(\sigma_4(X))$ by inheritance. Since it only involves partitions of length three, it cannot be generated by $I_5(\sigma_4(X))$, whose components all involve partitions of length four. Thus $I(\sigma_4(X))$ is not generated in degree 5.

- $I_6(\sigma_5(X)) = 0$
- $I_7(\sigma_5(X)) = 0$
- $I_8(\sigma_5(X)) \supseteq S_{5111}A \otimes S_{2222}B \otimes S_{2222}C \oplus S_{3311}A \otimes S_{2222}B \otimes S_{2222}C$, again, up to permutations. Both modules occur with multiplicity one in $S^8(A \otimes B \otimes C)$.
- $I_d(\sigma_6(X)) = 0$ for $d \leq 8$.

The factors $S_{311}A \otimes S_{2111}B \otimes S_{2111}C$ plus permutations in $I_5(\sigma_4(X))$ are inherited from case 6.8. Since $S_{2111}A \otimes S_{2111}B \otimes S_{2111}C$ is not in S^5V , all of $I_5(\sigma_4(X))$ must be inherited from case 6.8.

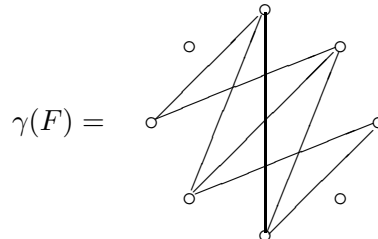
The remaining modules were eliminated by extensive Maple calculations. These calculations also showed us the candidate members of I_8 but only with extremely high probability, so we now present direct proofs that they are in the ideal.

The following monomial gives a highest weight vector for $S_{5111}A \otimes S_{2222}B \otimes S_{2222}C$ when summed over the symmetric group:

$$F = \alpha_1\alpha_2\alpha_5\alpha_6(\alpha_3\wedge\alpha_4\wedge\alpha_7\wedge\alpha_8)(\beta_1\wedge\beta_2\wedge\beta_3\wedge\beta_8)(\beta_4\wedge\beta_5\wedge\beta_6\wedge\beta_7)(\gamma_1\wedge\gamma_2\wedge\gamma_3\wedge\gamma_4)(\gamma_5\wedge\gamma_6\wedge\gamma_7\wedge\gamma_8).$$

A general element of $\sigma_5(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ is of the form $a_1b_1c_1 + \dots + a_5b_5c_5$, and when we compute a homogeneous polynomial P on such a sum, we get, after expansion, terms with different homogeneities on $a_1b_1c_1, \dots, a_5b_5c_5$. These homogeneous components must all vanish identically if we want P to vanish on $\sigma_5(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. Note that the type of homogeneity, up to permutation of 1...5, is given by a partition π with 5 parts, the sum of the parts being equal to the degree of the polynomial P .

We associate a graph $\gamma(F)$ to our tensor F . The vertices are identified with the integers 1...8, and two vertices i and j are joined by an edge iff they are not wedged together in the expression of F . We get:



Observe that $\gamma(F)$ contains *no triangle*, so that each time we choose a triple of indices among 1...8, two of them are wedged together somewhere in the expression of F .

Thus when we evaluate P on the monomials in the expansion of $(a_1b_1c_1 + \dots + a_5b_5c_5)^5$, all terms with a power of three or greater evaluate to zero.

There remains to consider the case where the degrees are $(2, 2, 2, 1, 1)$ (the case of $(2, 2, 2, 2)$ will follow). Denote the indices occuring with a power 2 by s, t, u and those to the first power by i, j . Note that s, t, u must appear twice in the contributions of A, B, C , but of course not in a same wedge product. So we'll only get terms of type

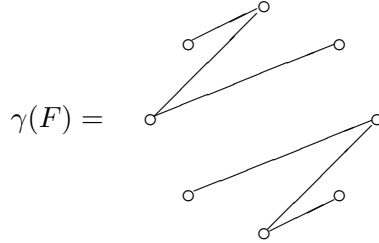
$$\alpha_s \alpha_t \alpha_\gamma \alpha_i (\alpha_s \wedge \alpha_t \wedge \alpha_u \wedge \alpha_j) (\beta_s \wedge \beta_t \wedge \beta_u \wedge \beta_i) (\beta_s \wedge \beta_t \wedge \beta_u \wedge \beta_j) (\gamma_s \wedge \gamma_t \wedge \gamma_u \wedge \gamma_i) (\gamma_s \wedge \gamma_t \wedge \gamma_u \wedge \gamma_j).$$

But this is skew-symmetric, e.g., in s and t , so the total contribution of these kinds of terms is zero.

For $S_{3311}A \otimes S_{2222}B \otimes S_{2222}C = S_{22}A \otimes \det A \otimes (\det B)^2 \otimes (\det C)^2$ the analysis is similar to the previous case. Here we may take

$$F = (\alpha_1 \wedge \alpha_3)(\alpha_5 \wedge \alpha_7)(\alpha_2 \wedge \alpha_4 \wedge \alpha_6 \wedge \alpha_8)(\beta_1 \wedge \beta_2 \wedge \beta_5 \wedge \beta_6)(\beta_3 \wedge \beta_4 \wedge \beta_7 \wedge \beta_8)(\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4)(\gamma_5 \wedge \gamma_6 \wedge \gamma_7 \wedge \gamma_8).$$

The associated graph is as follows. Again, it contains no triangle:



Finally, consider the terms we get with exponents $(2, 2, 2, 1, 1)$. They must be of type

$$(\alpha_s \wedge \alpha_t)(\alpha_u \wedge \alpha_i)(\alpha_s \wedge \alpha_t \wedge \alpha_u \wedge \alpha_j)(\beta_s \wedge \beta_t \wedge \beta_u \wedge \beta_i)(\beta_s \wedge \beta_t \wedge \beta_u \wedge \beta_j)(\gamma_s \wedge \gamma_t \wedge \gamma_u \wedge \gamma_i)(\gamma_s \wedge \gamma_t \wedge \gamma_u \wedge \gamma_j).$$

This is no longer skew-symmetric in s, t . But if we symmetrize with respect to s, t, u , we get (twice) the product of a fixed product of determinants, with $(\alpha_s \wedge \alpha_t)(\alpha_u \wedge \alpha_i) + (\alpha_t \wedge \alpha_u)(\alpha_s \wedge \alpha_i) + (\alpha_u \wedge \alpha_s)(\alpha_t \wedge \alpha_i)$. Since the vanishing of such an expression is precisely the condition that defines $S_{22}A$ inside $S^2(\wedge^2 A)$, our proof is complete.

In degree nine we verified that all cases of low multiplicity do not arise in $I(\sigma_6(X))$ and we are currently working on the cases of higher multiplicity. However, inspired by the $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ case, there are two natural candidates that we checked using $C++$ code written by P. Barbe:

Proposition 6.6. *The module*

$$S_{3333}A \otimes S_{3333}B \otimes S_{3333}C \subset S^{12}(A \otimes B \otimes C),$$

which occurs with multiplicity one, is not in $I(\sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$.

The polynomial in degree 12 may be obtained by symmetrizing

$$\begin{aligned} F(a_1b_1c_1, \dots, a_{12}b_{12}c_{12}) = & \det(a_1, a_2, a_3, a_4) \det(a_5, a_6, a_7, a_8) \det(a_9, a_{10}, a_{11}, a_{12}) \\ & \det(b_1, b_2, b_5, b_6) \det(b_3, b_7, b_9, b_{10}) \det(b_4, b_8, b_{11}, b_{12}) \\ & \det(c_1, c_7, c_9, c_{12}) \det(c_3, c_5, c_8, c_{10}) \det(c_2, c_4, c_6, c_{11}). \end{aligned}$$

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